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A straightforward method for finding implicit solitary wave solutions of nonlinear evolution and wave equations

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Abstract. We present a straightforward method for finding implicit solutions for nonlinear evolution and wave equations. The method is illustrated by finding implicit single solitary wave solutions for the Harry Dym, Korteweg-de Vries, modified Korteweg-de Vries, Boussinesq and the generalised Korteweg-de Vries equations.

1. Introduction

Explicit stationary travelling wave solutions of nonlinear dispersive evolution and wave equations can be derived using a variety of well known techniques. Notable among these are direct integration (wherever possible), the inverse scattering method (Ablowitz and Segur 1981), the Bäcklund transformation technique (Miura 1976), the Hirota method (Hirota 1980), 'perturbation' techniques (Sawada and Kotera 1974, Rosales 1978, Whitham 1979, Wadati and Sawada 1980a, b Hickernell 1983), the summation process of the Padé type (Turchetti 1980, Liverani and Turchetti 1983), direct linearisation techniques (Taflin 1983, Santini *et al* 1984), the Fredholm determinant method (Pöppe 1983, 1984) and the real exponential approach (Korpel 1978, Hereman *et al* 1985, 1986). For instance, when any of the above methods are applied to the Korteweg-de Vries (KdV) equation, one can readily derive the well known $\mathrm{sech}^2 K(x-vt)$-type solution, where v, the constant velocity of the hump-type solitary wave, is related to the width $1/K$. In fact, the real exponential approach has been employed to derive single solitary wave solutions of a large class of nonlinear evolution and wave equations. A comprehensive list of these equations and their solutions may be found in Hereman *et al* (1986).

However, in trying to derive a hump-type solution for the Harry Dym (HD) equation (Wadati *et al* 1979, 1980, Case 1982, Weiss 1983, Kawamato 1984a, b, Hereman *et al* 1989), it was found that no such solution could be obtained. All the equations listed in Hereman *et al* (1986) allow for solutions in terms of elementary functions (most often rational ones) of real exponentials, $\mathrm{e}^{K(x-vt)+\delta}$, where δ is a constant phase. The difficulty with the HD equation is that the phase is no longer constant but satisfies a transcendental equation. The presence of this transcendental phase gives rise to an implicit solution which when solved and plotted, resembles a cusp solitary wave (Hereman *et al* 1989).

In retrospect, the fact that nonlinear evolution and wave equations may have implicit solutions does not appear totally unnatural. For instance, recall that in the real exponential approach as originally introduced by Korpel (1978), the final solution for the nonlinear equation is assumed to be built up from the nonlinear mixings of the real exponential solutions to the linear dispersive part of the PDE. Alternatively, we may think of constructing a particular solution from the solution to the nonlinear non-dispersive part of the PDE. This is a valid conjecture, since the nonlinear non-dispersive part of the KdV equation in $u(x, t)$

$$u_t + uu_x = 0 \tag{1}$$

where the subscripts refer to the partial derivatives, possesses shock wave solutions (Whitham 1974) that are intrinsically implicit:

$$u(x, t) = g(x - u(x, t)t). \tag{2}$$

The implicit solution of the HD equation,

$$u_t = u^3 u_{3x} \tag{3}$$

can be written similarly as (Hereman *et al* 1989) $u(x, t) = F(f)$, with $f = x - vt + G(f)$, and where $G_f = 1 - F$.

Based on these two examples and on the discussion above, we may think of solutions to an arbitrary nonlinear dispersive PDE to be of the form

$$u(x, t) = F(f) \tag{4a}$$

with

$$f(x, t) = H_1(f)x - H_2(f)t + H_3(f) \tag{4b}$$

where $f(x, t)$ may be regarded as a Riemann invariant while the implicit solution u is what has been known as the Riemann wave (Whitham 1974, Kalinowski 1982).

It is true that the HD equation is different from other nonlinear dispersive evolution equations (namely the KdV: $u_t + \alpha uu_x + u_{3x} = 0$) in the sense that it does not possess a linear dispersive part. Is it true, therefore, that this feature ensures that its solution is an implicit one, since no implicit solutions of equations like the KdV equation have been reported? We have, on the basis of our examination of some nonlinear evolution and wave equations, found the answer to be negative. One may be led to argue that the implicit nature of the solutions to the HD and the kinematic wave equations is due to the existence of a hodograph transformation, involving a change of dependent and independent variables, which transforms the equations into explicit solvable ones. For instance, the HD can be transformed into the modified KdV (MKdV) equation (Hereman *et al* 1989) as follows. Using the hodograph transformation

$$X = \int_{-\infty}^{x} \frac{\mathrm{d}s}{u(s, t)} \tag{5}$$

equation (3) can be recast into the auxiliary equation

$$R_t - R_{3X} - (3R_X/R^2)(\tfrac{1}{2}R_X^2 - RR_{2X}) = 0 \tag{6}$$

for $R(X(x, t), t) = u(x, t)$, where the new independent variable X depends on x and t through the old dependent variable u. By the Cole-Hopf transformation $R = \Gamma_x/\Gamma$, (6) can then be further reduced to the MKdV in Γ.

Along the same vein, (1) can be 'linearised' into

$$X_t = u \tag{7}$$

with

$$x = X(u, t) = \int_{-\infty}^{x} u(x, s)\,\mathrm{d}s. \tag{8}$$

Incidentally, the hodograph transformations (5) and (8) also cause decoupling of the nonlinearity from dispersion. Inversion of the hodograph transformations clearly make the explicit solutions of (6) and (7) implicit.

In this paper we investigate the possibility of constructing implicit solitary wave solutions to some integrable PDEs, e.g. the KdV, the MKdV and the Boussinesq (BE) equations. A brief discussion on the nature of these solutions, the role of dispersion, the significance of such implicit solutions and general speculation on whether these solutions could have been obtained using the real exponential method is now in order. We remind readers that the implicit nature of the solutions to the HD equation and the kinematic wave equation comes from the hodograph transformation as explained in the previous paragraph. Furthermore, if the implicit solution of the HD equation is retransformed hodographically to a possible solution of the KdV (or MKdV), the result is an explicit solution of the latter equation containing a mixture of exponential and rational forms. However, the solutions of the KdV, the MKdV and the BE which we will present below are inherently implicit, and different from both the well known explicit solutions derivable from classical inverse scattering or direct integration and the rational-exponential explicit solutions obtainable from the implicit solution of the HD.

It is worthwhile to note that the role of dispersion, as projected in conventional physical pictures of solitary wave formation, is now somewhat different. The traditional picture portrays nonlinearity to cause steepening of a (baseband) pulse and dispersion to cause spreading, resulting in a smooth hump-type solution which remains unchanged in shape as it travels. From a more relaxed viewpoint, we can visualise the dispersion in, for instance, the KdV, as being instrumental in preserving the shape of the pulse, which would otherwise have continually steepened from the action of the nonlinearity alone till the advent of shock. The latter is portrayed by the solution to the kinematic wave equation (see (2)).

We also remark that if we restrict ourselves to implicit single solitary wave solutions, integrability is not an essential factor since it is possible to apply our method to non-integrable versions of the BE, namely the improved and the modified improved Boussinesq equations (Iskandar and Jain 1980, Soerensen *et al* 1982).

The organisation of the paper is as follows. In section 2, we develop the solution method taking (4) as our starting point. We then use it for the HD equation as our first example (section 3). In section 4 we obtain a new solution for the KdV equation which is then checked numerically by putting it in as an initial condition and thereafter monitoring its propagation. This is then followed up by examples constituting the MKdV equation (section 5) and the BE equation (section 6). In section 7, we demonstrate the applicability of our solution method to non-integrable systems. The example considered will be the generalised KdV equation for $n=4$. Finally, in section 8, conditions for the existence of implicit solutions, including the conditions on the functions H_1 and H_2, are specified. A list of the nonlinear PDEs solved using our technique, and their implicit solutions is also presented.

2. The solution method

The procedure for attempting to find implicit solitary wave solutions of nonlinear PDEs may be summarised in the following steps.

(1) We start from the general form suggested in (4) and rewrite the given equation as a differential equation for $F(f)$. The coefficients in this ODE will include H_1, H_2 and H_3 and their derivatives with respect to f. This is achieved by replacing $\partial/\partial t$ and

$\partial/\partial x$ in terms of $\partial/\partial f$ as

$$\partial/\partial t = f_t \partial/\partial f \qquad \partial/\partial x = f_x \partial/\partial f \tag{9a}$$

and subsequently calculating f_t and f_x from (4)

$$f_t = -H_2/D \qquad f_x = H_1/D \tag{9b}$$

with

$$D(f) = 1 - H_{1,f}x + H_{2,f}t - H_{3,f}. \tag{9c}$$

(2) We then have to carry out the integration(s) until we find the solution for F terms of f. This will impose a restriction on some of the Hs and requires an appropriate choice for D. Since we are only interested in stationary travelling wave solutions that do not change their shape, we have to set

$$H_2 = vH_1 \tag{10}$$

where v is the velocity of the travelling wave. Also, as will be evident from the examples in the following sections, the final step will usually entail an expression of the form

$$\mathrm{d}F/\mathrm{d}f = (D/H_1)F^{\eta}\{P(F)\}^{1/2} \tag{11}$$

where η is a constant and $P(F)$ is a polynomial in F. The crux of the method for finding implicit solutions lies in choosing D/H_1 to be an appropriate explicit function of F rather than of f. The reason for this will become clear below.

(3) After we have found the solution F, we have to determine the implicit variable f and its relation with x and t. We will start from (9c) and (10) by expressing $(x - vt)$ in terms of H_1, H_3 and D as

$$(x - vt) = [1 - D - H_{3,f}]/H_{1,f}. \tag{12}$$

Substituting in (4b) with (10), we get

$$H_{3,f} - (H_{1,f}/H_1)H_3 = 1 - (H_{1,f}/H_1)f - D \tag{13}$$

which, upon division by H_1, may be integrated to give

$$H_3(f) = f - H_1(f)\int (D/H_1)\,\mathrm{d}f + CH_1(f) \tag{14}$$

where C is an integration constant. By choosing an appropriate function of f for H_1, we can solve for H_3.

Note that D/H_1 may not be chosen as a function of f. If this choice were made, (11) may be re-expressed as $\int (D/H_1)\,\mathrm{d}f = \int \mathrm{d}F/[F^{\eta}\{P(F)\}^{1/2}]$, enabling F to be expressed as a function of $\int (D/H_1)\,\mathrm{d}f$ after integration. But, from (4b) and (14) with $C = 0$, it readily follows that F would be an explicit function of $(x - vt)$. For the KdV, MKdV and BE equations, the well known hump-type solutions are then readily recovered.

For the rest of the paper we will tacitly assume that D/H_1 is an explicit function of F.

(4) Finally, knowing the implicit solution (F and H_3 as functions of f) and f as a function of x and t as in (4b), we can plot the explicit solution u against x and t.

3. Example 1: the Harry Dym equation

To make this paper self-contained, as well as to convince readers of the applicability of our methodology outlined above, we will rederive the implicit single solitary wave solution of the HD equation (3) (Hereman *et al* 1989).

In accordance with step (1), we first rewrite (3) entirely in terms of f. To achieve this, we use (4) and (9) and obtain

$$-H_2F_f = F^3H_1[(\partial/\partial f)(H_1/D)[(\partial/\partial f)(H_1/D)F_f]]. \tag{15}$$

As a second step in finding single solitary wave solutions, we use (10), and integrate (15) twice:

$$\mathrm{d}F/\mathrm{d}f = (D/H_1)\{-2c_1F - 2c_2 + vF^{-1}\}^{1/2} \tag{16}$$

which is indeed of the form of (11). The quantities c_1 and c_2 are integration constants. Now, we make the appropriate choice for D/H_1, namely

$$D/H_1 = F \tag{17}$$

and set $v = c_2 = -2c_1$. One more integration then yields

$$F(f) = \tanh^2\{(v/4)^{1/2}f\} \tag{18}$$

where v has to be positive.

The third step involves the evaluation of $H_3(f)$. Using (17) and (18) in (14), we can write

$$H_3(f) = (1 - H_1(f))f + CH_1(f) + H_1(f)(4/v)^{1/2}\tanh\{(v/4)^{1/2}f\}. \tag{19}$$

The functions $F(f)$ and $H_3(f)$ are plotted in figures 1(*a*) and (*b*) respectively, for $H_1 = \text{constant} = \frac{1}{2}$, $v = 2$ and $C = 0$. Figures 1(*c*) and (*d*) show $u(x, t)$ and $\tilde{H}_3\,(x, t) = H_3(f)$ against x, at $t = 0$, and were plotted in accordance with step 4 of the general procedure. Our result is similar to the solution reported by Hereman *et al* (1989).

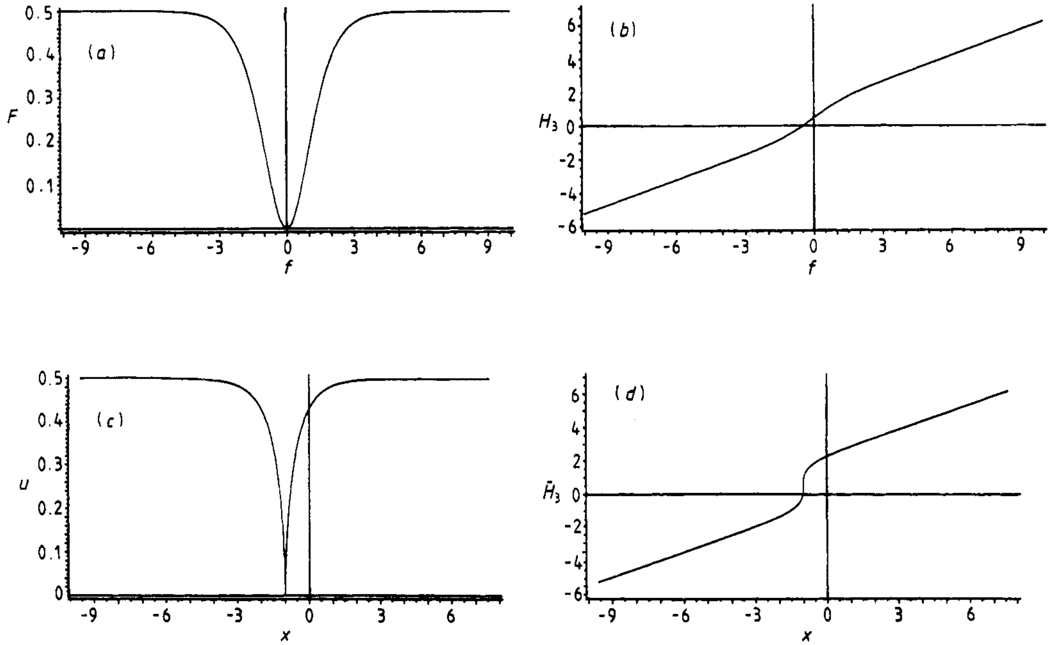

Figure 1. The implicit solution for the HD equation with $H_1(f) = \frac{1}{2}$, $v = 2$ and $C = 0$. (*a*) F (equation (18)) against f; (*b*) H_3 (equation (19)) against f; (*c*) $u(x, t)$ against x at $t = 0$; (*d*) $\tilde{H}_3(x, t)$ against x at $t = 0$.

4. Example 2: the Korteweg–de Vries equation

As a second example to show the implementation of implicit solutions we have chosen the KdV equation (Korteweg and de Vries 1895, Lamb 1980, Hereman *et al* 1986)

$$u_t + \alpha u u_x + u_{3x} = 0 \tag{20}$$

where α is a nonlinearity constant, and where the coefficient of the dispersive term u_{3x} has been scaled to unity.

Combining (4), (9) and (10), u and its derivatives are expressible as

$$\begin{aligned}
&u(x,t) \rightarrow F(f)\\
&u_t \quad \rightarrow -v(H_1/D)f_f\\
&u_x \quad \rightarrow (H_1/D)F_f\\
&u_{2x} \quad \rightarrow (H_1/D)(\partial/\partial f)[(H_1/D)F_j]\\
&u_{3x} \quad \rightarrow (H_1/D)(\partial/\partial f)[(H_1/D)(\partial/\partial f)[(H_1/D)F_f]]
\end{aligned} \tag{21a}$$

with

$$D(f) = 1 - H_{1,f}(x - vt) - H_{3,f}. \tag{21b}$$

With the above substitutions, (20) reads

$$-vF_f + \alpha F F_f + (\partial/\partial f)[(H_1/D)(\partial/\partial f)[(H_1/D)F_f]] = 0. \tag{22}$$

Hence, upon two integrations, (22) becomes

$$\mathrm{d}F/\mathrm{d}f = D/H_1\{-(\alpha/3)F^3 + vF^2 + 2c_1F + 2c_2\}^{1/2} \tag{23}$$

where c_1 and c_2 are integration constants. Choosing $c_2 = 0$ and $D/H_1 = (-F)^{1/2}$ for convenience, (23) is readily integrated (Gradshteyn and Ryzhik 1984) to obtain

$$F(f) = \frac{-b \pm [(4ac - b^2)\tanh^2(\rho)/(1 - \tanh^2(\rho))]^{1/2}}{[b^2 - 4ac\tanh^2(\rho)]/[2a(1 - \tanh^2(\rho))]} \tag{24a}$$

with

$$c = \alpha/3 \qquad b = -v \qquad a = -2c_1 > 0 \qquad b^2 \leqslant 4ac \qquad \rho = (a)^{1/2}f. \tag{24b}$$

Since we have the solution for $F(f)$, (14) gives a relationship between H_1 and H_3. For the particular case where H_1 is constant we would have

$$H_3(f) = f - H_1 \int (-F)^{1/2}\,\mathrm{d}f + CH_1. \tag{25}$$

Figures 2(a) and (b) show both F and H_3 for $H_1 = a = b(= -v) = c = 1$ and $C = 0$ as functions of f, while figures 2(c) and (d) show $u(x,t)$ and $\tilde{H}_3(x,t) = H_3(f)$ as functions of $x - vt$ at $t = 0$. H_3 is numerically computed using (14), (23) and (24). Thereafter, $x - vt$ is computed as a function of f using (4b) and (10), and combined with figures 2(a) and (b) to generate figures 2(c) and (d).

In order to be absolutely sure that we have, in fact, found a new solution, we program the KdV equation (20) with the initial condition as in figure 2(c). A finite difference scheme with proper modification to ensure stability of the numerical algorithm, as suggested by Dodd *et al* (1982), is employed. This demands ensuring that $\Delta t/(\Delta x)^3 \leqslant (4 + (\Delta x)^2|u_0|)^{-1}$ where Δt, Δx are the time and space step sizes and

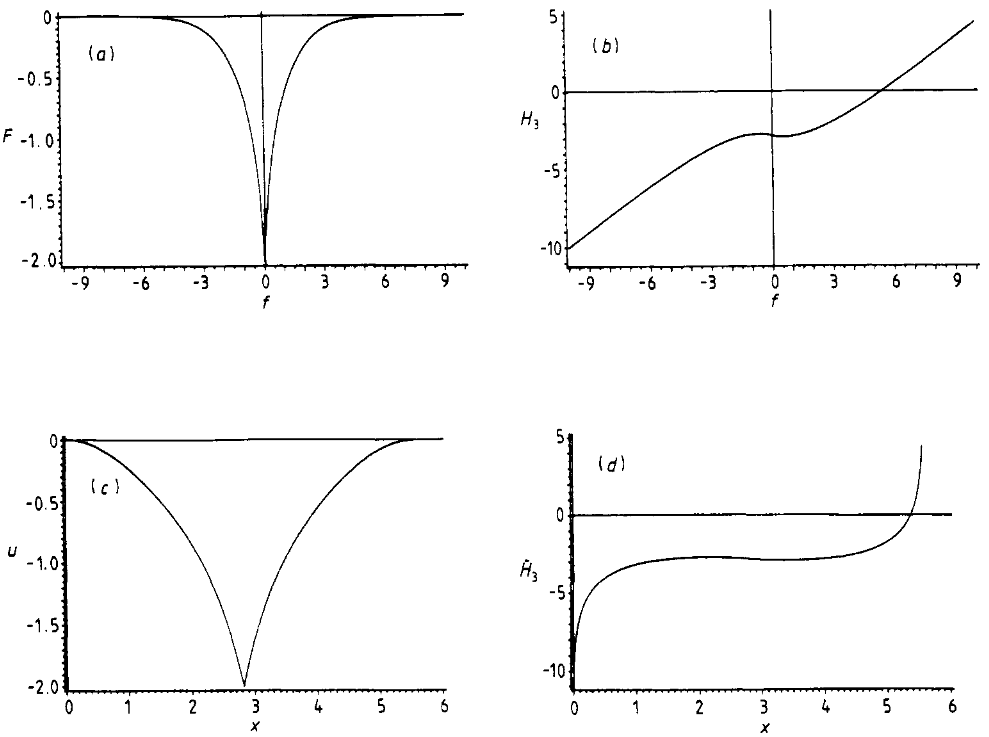

Figure 2. A new implicit solution for the KdV equation $H_1(f)=1$, $a, b, c=1$ and $C=0$. (*a*) F (equation (24)) against f; (*b*) H_3 (equation (25)) against f; (*c*) $u(x,t)$ against x at $t=0$; (*d*) $\tilde{H}_3(x,t)$ against x at $t=0$.

u_0 is the maximum value of u over the range of interest. Note that (20) has been written in a moving frame of reference with a velocity c_0 which, though explicitly absent from (20) and, hence, from the program, implicitly comes in through the ratio $\Delta x/\Delta t$. The computational advantage in programming (20) in the travelling frame lies in the fact that a much smaller grid size may be used. Figure 3 shows the propagation of the initial condition as in figure 2(*c*) over $t=3.33\times10^{-3}$. With the choice of $\Delta x=2.83\times10^{-3}$ and $\Delta t=5.553\times10^{-9}$; c_0 becomes equal to 509 637.11, corresponding to a translation of 1698 in the laboratory frame of reference. The figures have been drawn in the laboratory frame of reference to explicitly bring out the preservation of the waveshape after a distance 1698 of travel, which corresponds to about 566 times the width of the initial pulse. Figure 4 shows the distortion after propagation for an initial condition $2u(x,0)$ with $u(x,0)$ as in figure 2(*c*). An initial condition $\frac{1}{2}u(x,0)$ also shows similar distortion after the same distance of propagation.

5. Example 3: the modified Korteweg–de Vries equation

The MKdV equation (Lamb 1980, Dodd *et al* 1982) is quite similar to the KdV but has a cubic nonlinearity. Both equations are connected by the Miura transformation (Lamb 1980). If u is a solution to the MKdV equation

$$u_t+\alpha u^2u_x+u_{3x}=0 \tag{26}$$

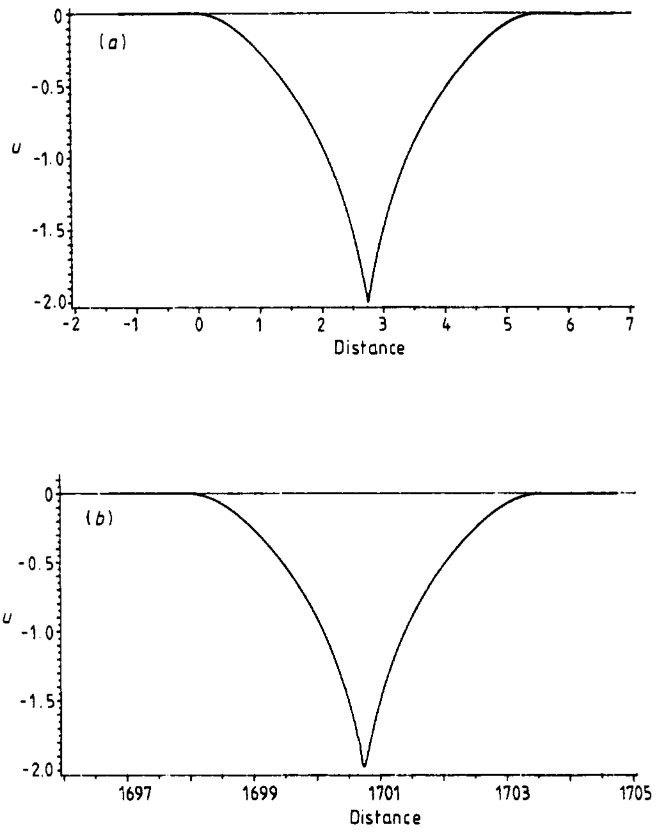

Figure 3. Propagation of the implicit solution for the KdV equation for $H_1(f) = 1$, $a = b = c = 1$ and $C = 0$ at (*a*) $t = 0$, (*b*) $t = 3.33 \times 10^{-3}$. The horizontal axis represents distance in a laboratory frame of reference.

then

$$w = \alpha(u^2 + (-6/\alpha)^{1/2} u_x)/\alpha_1 \tag{27}$$

is a solution to the KdV equation

$$w_t + \alpha_1 w w_x + w_{3x} = 0. \tag{28}$$

As in the KdV case, we use substitutions as in (21) to rewrite (26) as:

$$-vF_f + \alpha F^2 F_f + (\partial/\partial f)[(H_1/D)(\partial/\partial f)[(H_1/D)F_f]] = 0 \tag{29}$$

After three integrations, (29) becomes

$$f = \int (D/H_1)^{-1}\{-(\alpha/6)F^4 + vF^2 + 2c_1 F + 2c_2\}^{-1/2}\,\mathrm{d}F. \tag{30}$$

We now introduce a new function G such that

$$F = (-G)^{1/2} \tag{31}$$

and select

$$D/H_1 = F/2. \tag{32}$$

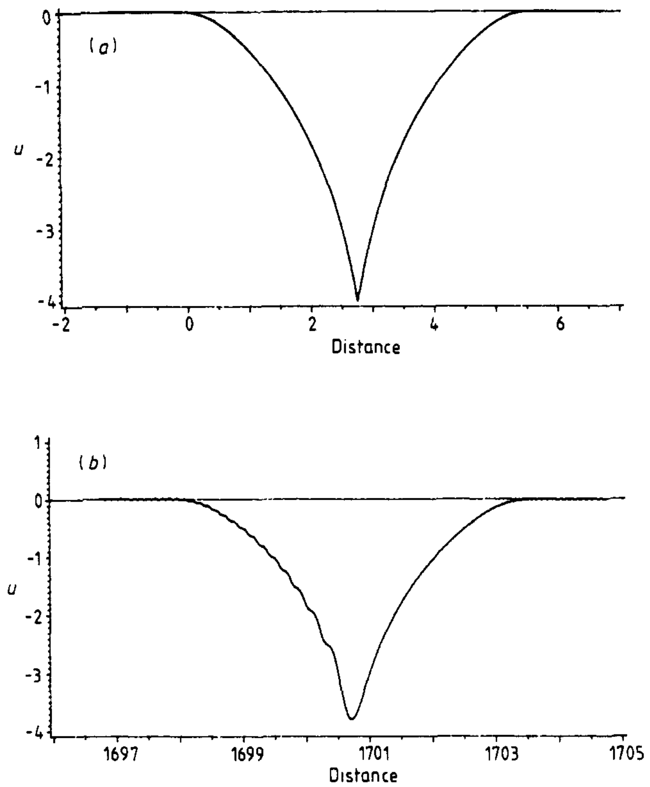

Figure 4. Propagation of an initial condition equal to twice the KdV solution shown in figure 2(*c*) at (*a*) $t=0$, (*b*) $t=3.33\times10^{-3}$. The horizontal axis has the same meaning as in figure 3.

With these assumptions, and upon setting $c_1=0$, (30) becomes

$$f=-\int G^{-1}\{-(\alpha/6)G^2-vG+2c_2\}^{-1/2}\,\mathrm{d}G. \tag{33}$$

As may be readily verified the solution for G is expressible as (Gradshteyn and Ryzhik 1984)

$$G(f)=\frac{-b+[(4ac-b^2)\tanh^2(\rho)/(1-\tanh^2(\rho))]^{1/2}}{[b^2-4ac\tanh^2(\rho)]/[2a(1-\tanh^2(\rho))]} \tag{34a}$$

with

$$c=-\alpha/6 \qquad b=-v \qquad a=2c_2>0 \qquad \rho=-(a)^{1/2}f. \tag{34b}$$

The solution to (29) then finally is

$$F(f)=\left(\frac{b-[(4ac-b^2)\tanh^2(\rho)/(1-\tanh^2(\rho))]^{1/2}}{[b^2-4ac\tanh^2(\rho)]/[2a(1-\tanh^2(\rho))]}\right)^{1/2} \tag{35}$$

while H_3 from (14), upon taking $H_1(f)=f$ for variety, is

$$H_3(f)=(C+1)f-f\int(F/2)\,\mathrm{d}f. \tag{36}$$

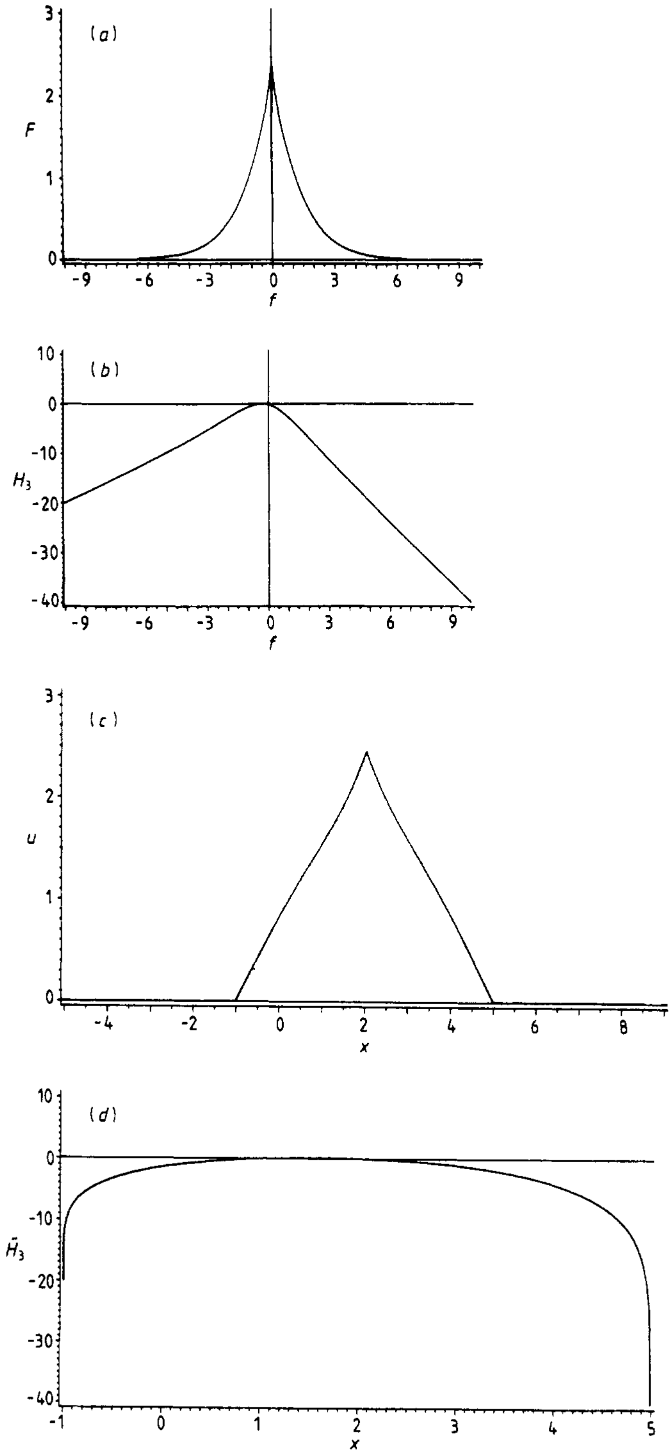

Figure 5. A new implicit solution for the MKdV equation with $H_1(f)=f$, $a=3$, $b=1$, $c=0.25$ and $C=1$. (a) F (equation (35)) against f; (b) H_3 (equation (36)) against f; (c) $u(x, t)$ against x at $t=0$; (d) $\bar{H}_3(x, t)$ against x at $t=0$.

Figures 5(*a*), (*b*) and (*c*), (*d*) show F and H_3 for $H_1(f)=f$, $a=3$, $b=1$, $c=0.25$ and $C=1$ as functions of f and, u, $\tilde{H}_3$ as functions of $x-vt$ at $t=0$, respectively.

Straightforward application of the Miura transformation will lead us to yet another solution to the KdV equation.

6. Example 4: the Boussinesq equation

As an example of a wave equation we choose the BE equation which was first derived by Boussinesq (1871, 1872) to describe shallow-water waves propagating in both directions. It has been also used to describe displacements in a one-dimensional lattice with an exponential potential (Zabusky 1967). The assumed form for the BE equation will be

$$u_{2t}-u_{2x}-u_{4x}+3\alpha(u^2)_{2x}=0. \tag{37}$$

Adhering to the strategy of the method, we involve (21) in (37) to give

$$\begin{aligned} &v^2(\partial/\partial f)[(H_1/D)F_f]-(\partial/\partial f)[(H_1/D)F_f] \\ &\quad -(\partial/\partial f)[(H_1/D)(\partial/\partial f)[(H_1/D)(\partial/\partial f)[(H_1/D)F_f]]] \\ &\quad +6\alpha(\partial/\partial f)(H_1/D)FF_f=0. \end{aligned} \tag{38}$$

After two integrations, we obtain

$$(v^2-1)F-(H_1/D)(\partial/\partial f)[(H_1/D)F_f]+3\alpha F^2=c_1f+c_2 \tag{39}$$

where c_1, c_2 are integration constants. Choosing $c_1=0$, then multiplying by F_f and next integrating for a third time, results in

$$\tfrac{1}{2}(v^2-1)F^2-\tfrac{1}{2}[(H_1/D)F_f]^2+\alpha F^3=c_2F+c_3 \tag{40}$$

where c_3 is another integration constant.

Following the same steps as in section 4 we end up with the same answer as for the KdV:

$$F(f)=\frac{-b\pm[(4ac-b^2)\tanh^2(\rho)/(1-\tanh^2(\rho))]^{1/2}}{[b^2-4ac\tanh^2(\rho)]/[2a(1-\tanh^2(\rho))]} \tag{41a}$$

but with

$$c=2\alpha \qquad b=(v^2-1) \qquad a=-2c_2>0 \qquad \rho=(a)^{1/2}f \tag{41b}$$

and

$$H_3(f)=f-H_1(f)\int(F)^{1/2}\,\mathrm{d}f+CH_1(f). \tag{42}$$

With $H_1=a=b=C=1$ and $c=2$, the plots for the BE equation become identical to figures 2(*a*), (*b*), (*c*) and (*d*) drawn for the KdV equation.

7. Example 5: the generalised KdV equation

The purpose of this section will be to demonstrate the applicability of our solution method to non-integrable systems. An example considered here is the generalised KdV equation (Lax 1968)

$$u_t+\alpha u^n u_x+u_{3x}=0. \tag{43}$$

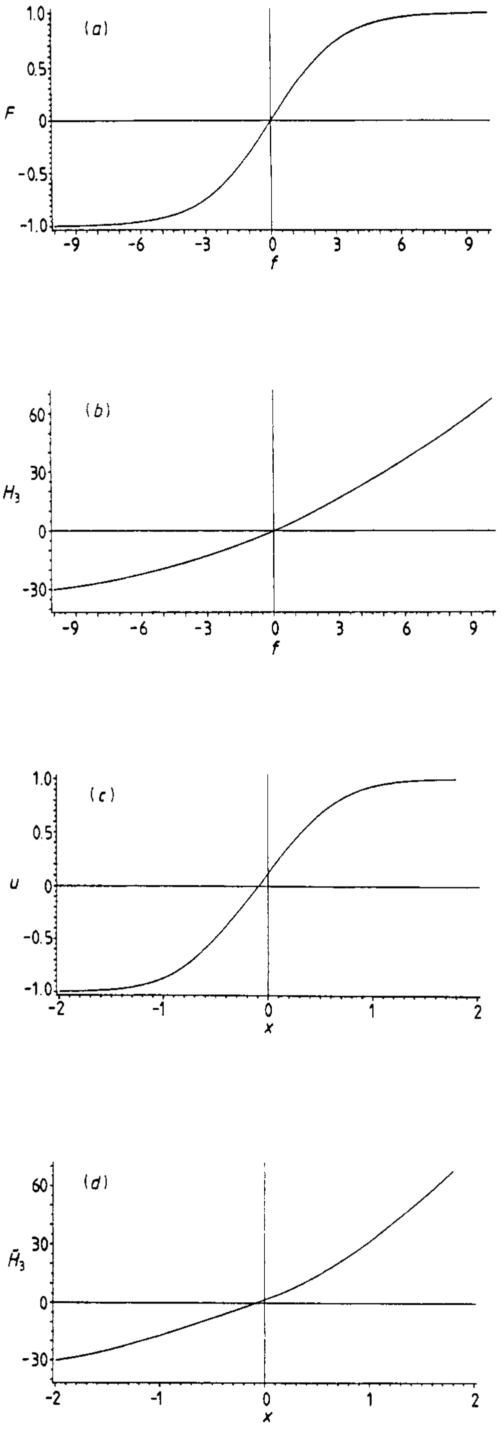

Figure 6. A new implicit solution of the generalised KdV equation for $n=4$ and with $H_1(f)=f$, $\alpha=-6$, $v=-1.2$ and $C=2$. (*a*) F (equation (50)) against f; (*b*) H_3 (equation (51)) against f; (*c*) $u(x,t)$ against x at $t=0$; (*d*) $\tilde{H}_3(x,t)$ against x at $t=0$.

The particular cases of $n=1$ (KdV) and $n=2$ (MKdV), which are known to be integrable, have already been solved in sections 4 and 5. For all other values of n, the equation (43) is non-integrable and possesses a finite number of conserved quantities (Lax 1968). Using the same set of transformations as before (see (23)), equation (43) becomes

$$-vF_f+\alpha F^nF_f+(\partial/\partial f)[(H_1/D)(\partial/\partial f)[(H_1/D)F_f]]=0. \tag{44}$$

After three integrations, (44) becomes

$$f=\int (D/H_1)^{-1}\{-[2\alpha/(n+1)(n+2)]F^{n+2}+vF^2+2c_1F+2c_2\}^{-1/2}\,\mathrm{d}F. \tag{45}$$

To readily get closed-form results, we choose $c_1=0$, $c_2=-\alpha/15$, $v=\alpha/5$ and $n=4$. Now, by introducing a new function G such that

$$F=(G)^{1/2} \tag{46}$$

and selecting

$$D/H_1=1/(4+2G) \tag{47}$$

equation (45) becomes

$$f=\int G^{-1/2}\{(-v/3)(G-1)^2\}^{-1/2}\,\mathrm{d}G. \tag{48}$$

As may be readily verified the solution for G is expressible as (Gradshteyn and Ryzhik 1984)

$$G=\tanh^2\{(-v/12)^{1/2}f\} \qquad v<0. \tag{49}$$

The solution to (44) then finally is

$$F=\tanh\{(-v/12)^{1/2}f\} \qquad v<0 \tag{50}$$

while H_3 from equation (16), upon taking $H_1(f)=f$, is

$$H_3(f)=(C+1)f-f\int (D/H_1)\,\mathrm{d}f. \tag{51}$$

Figure 6(a), (b) and (c), (d) show F and H_3 for $H_1(f)=f$, $\alpha=-6$, $v=-1.2$ and $C=1$ as functions of f and u, $\tilde{H}_3$ as functions of $x-vt$ at $t=0$, respectively.

8. Discussion and conclusion

Through the above examples of the HD, KdV, MKdV, BE and the generalised KdV equations we have shown the simplicity and the ease of the method for finding implicit solutions. We may remark that the integrability of the PDE is not essential for the existence of implicit solutions. For instance, the non-integrable modifications of the BE (e.g. the improved Boussinesq and the modified improved Boussinesq equations (Iskandar and Jain 1980, Soerensen *et al* 1982) may be shown to possess implicit solitary wave solutions similar to that of the BE. This is because the resulting ODE, after the change of variables to a travelling frame of reference, is similar to equation (39).

Table 1. A list of the PDEs solved using the implicit formalism, the choice of the ratio D/H_1 and the expressions for $F(f)$ and $H_3(f)$.

Partial differential equations	D/H_1	$F(f)$	$H_3(f)$
KdV			
$u_t+\alpha uu_x+u_{3x}=0$	$(-F)^{1/2}$	$F(f)=-b\pm[(4ac-b^2)\tanh^2(\rho)/(1-\tanh^2(\rho))]^{1/2}$ $\times\{[b^2-4ac\tanh^2(\rho)]/[2a(1-\tanh^2(\rho))]\}^{-1}$ with $c=\alpha/3,\ b=-v,\ a=-2c_1>0,\ b^2\leqslant 4ac,\ \rho=(a)^{1/2}f$	$H_3(f)=f-H_1\int(-F)^{1/2}\,df+CH_1$
mKdV			
$u_t+\alpha u^2u_x+u_{3x}=0$	F	$F(f)=\{b-[(4ac-b^2)\tanh^2(\rho)/(1-\tanh^2(\rho))]\}^{1/4}$ $\times\{[b^2-4ac\tanh^2(\rho)]/[2a(1-\tanh^2(\rho))]\}^{-1/2}$ with $c=-\alpha/6,\ b=v,\ a=2c_2>0,\ \rho=-(a)^{1/2}f$	$H_3(f)=(C+1)f-f\int F\,df$
Generalised KdV equation $(n=4)$			
$u_t+\alpha u^4u_x+u_{3x}=0$	$1/(4+2F^2)$	$F=\tanh\{(-v/12)^{1/2}f\}\quad \alpha=5/v<0$	$H_3(f)=(C+1)f-f\int(D/H_1)\,df$
BE			
$u_{2t}-u_{2x}-u_{4x}+3\alpha(u^2)_{2x}=0$	$(F)^{1/2}$	$F(f)=-b\pm[(4ac-b^2)\tanh^2(\rho)/(1-\tanh^2(\rho))]^{1/2}$ $\times\{[b^2-4ac\tanh^2(\rho)]/[2a(1-\tanh^2(\rho))]\}^{-1}$ with $c=2\alpha,\ b=-(1-v^2),\ a=-2c_2>0,\ \rho=(a)^{1/2}f$	$H_3(f)=f-H_1\int(F)^{1/2}\,df+CH_1$
Improved BE			
$u_{2t}-u_{2x}+\alpha u_x^2+\alpha uu_{2x}+\delta u_{2x2t}=0$	$(F)^{1/2}$	$F(f)=-b\pm[(4ac-b^2)\tanh^2(\rho)/(1-\tanh^2(\rho))]^{1/2}$ $\times\{[b^2-4ac\tanh^2(\rho)]/[2a(1-\tanh^2(\rho))]\}^{-1}$ with $c=2\alpha/v^2,\ b=(v^2-1)/v^2,\ a=-2c_2/v^2>0,\ \rho=(a)^{1/2}f$	$H_3(f)=f-H_1\int(F)^{1/2}\,df+CH_1$
Modified improved BE			
$u_{2t}-u_{2x}+2\alpha uu_x^2+\alpha u^2u_{2x}-\delta u_{2x2t}=0$	F^2	$F(f)=\{b-[(4ac-b^2)\tanh^2(\rho)/(1-\tanh^2(\rho))]\}^{1/4}$ $\times\{[b^2-4ac\tanh^2(\rho)]/[2a(1-\tanh^2(\rho))]\}^{-1/2}$ with $c=\alpha/v^2,\ b=(v^2-1)/v^2,\ a=-2c_2/v^2>0,\ \rho=(a)^{1/2}f$	$H_3(f)=f-H_1\int F^2\,df+CH_1$
HD equation			
$u_t=u^3u_{3x}$.	F	$F(f)=\tanh^2\{(v/4)^{1/2}f\}$	$H_3(f)=(1-H_1(f))f+CH_1(f)+H_1(f)(4/v)^{1/2}\tanh\{(v/4)^{1/2}f\}$
Generalised HD equation $(n=4)$			
$u_t=u^4u_{3x}$	$[F^3/(1+2F^2)]^{1/2}$	$F(f)=\tanh^2\{(-v/12)^{1/2}f\}$ $v<0$	$H_3(f)=f-H_1\int[F^3/(1+2F^2)]^{1/2}\,df+CH_1$

The effectiveness of the method is limited by the class of integrals expressible in closed form which, in turn, imposes a severe restriction on the degree of nonlinearity in the PDE. For instance the generalised HD equation $u_t = u^n u_{3x}$ may be shown to have non-physical solutions for $n = 1$ and 2. For $n = 4$, a $\tanh^2$-type solution for $F(f)$ is possible through a clever choice of $D/H_1 F(F/(1+2F))^{1/2}$. For $n > 4$, it is not possible to obtain closed-form solutions. Similarly, in the class of generalised KdV equations $u_t + \alpha u^n u_x + u_{3x} = 0$, closed-form solutions are obtainable for $n = 4$ over and above the cases $n = 1$ (KdV) and $n = 2$ (MKdV) discussed in the paper. Specifically, for $n = 4$, the choice $D/H_1 = \frac{1}{2}(2 + F^2)$ yields a tanh-type solution for $F(f)$, with proper choices for some of the integration constants. Again, for $n = 3$ and $n > 4$, no closed form solutions appear to be possible.

Notwithstanding these limitations, it must be reiterated that the implicit solutions derived in this paper for the KdV, MKdV and BE equations are new and not merely the previously known hump-type solitary wave solutions in disguise. It is clear from the discussion in the introduction that the implicit solution to, for instance, the KdV equation, is inherently different from that of the HD equation or the solution of the latter transformed hodographically. Furthermore, conventional solutions of the KdV, MKdV and BE equations are obtainable only by choosing D/H_1 as a function of f rather than F. Moreover, the solutions of the above equations, when plotted, are cusp-type and different from the conventional sech- or sech^2-type solutions. Finally, when allowed to propagate in accordance to their respective equations, the solutions show no change in shape.

Table 1 lists the PDEs solved using our implicit formalism, the choice of the ratio D/H_1 and the expressions for $F(f)$ and $H_3(f)$. Incorporating equation (4*b*), $u(x, t)$ and $H_3(x, t)$ can be determined.

Finally, we would like to mention that further work is being done to employ the technique developed in the paper for more complicated examples including coupled systems, and for cases where the velocity, given as the ratio of H_2/H_1, is not constant but rather a function of F.

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